



UNIVERSITÉ
DE GENÈVE

Decision Making Under Uncertainty

Week 6

Bayes Rule. Belief update
Probability judgment errors

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Uncertainty (and how to integrate information)

They hear a noise and are uncertain. flight? or eat?

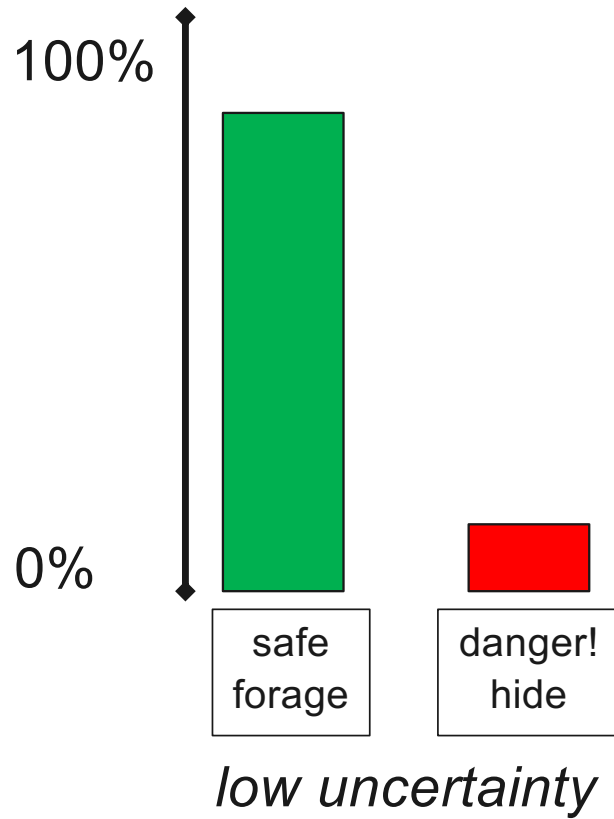
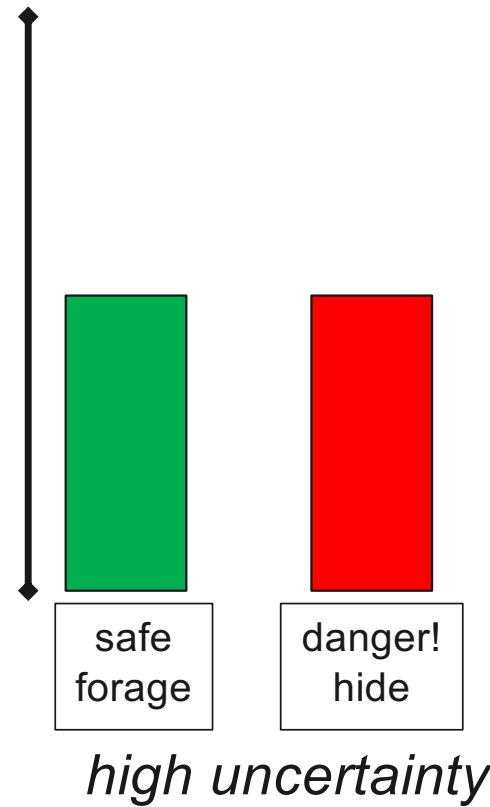
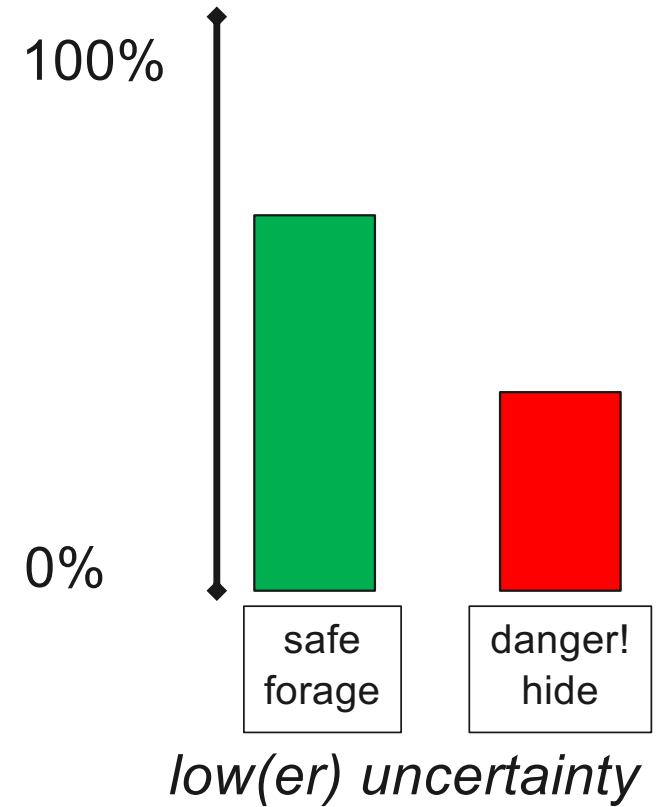
These meerkats are collecting more evidence (to remove the uncertainty)



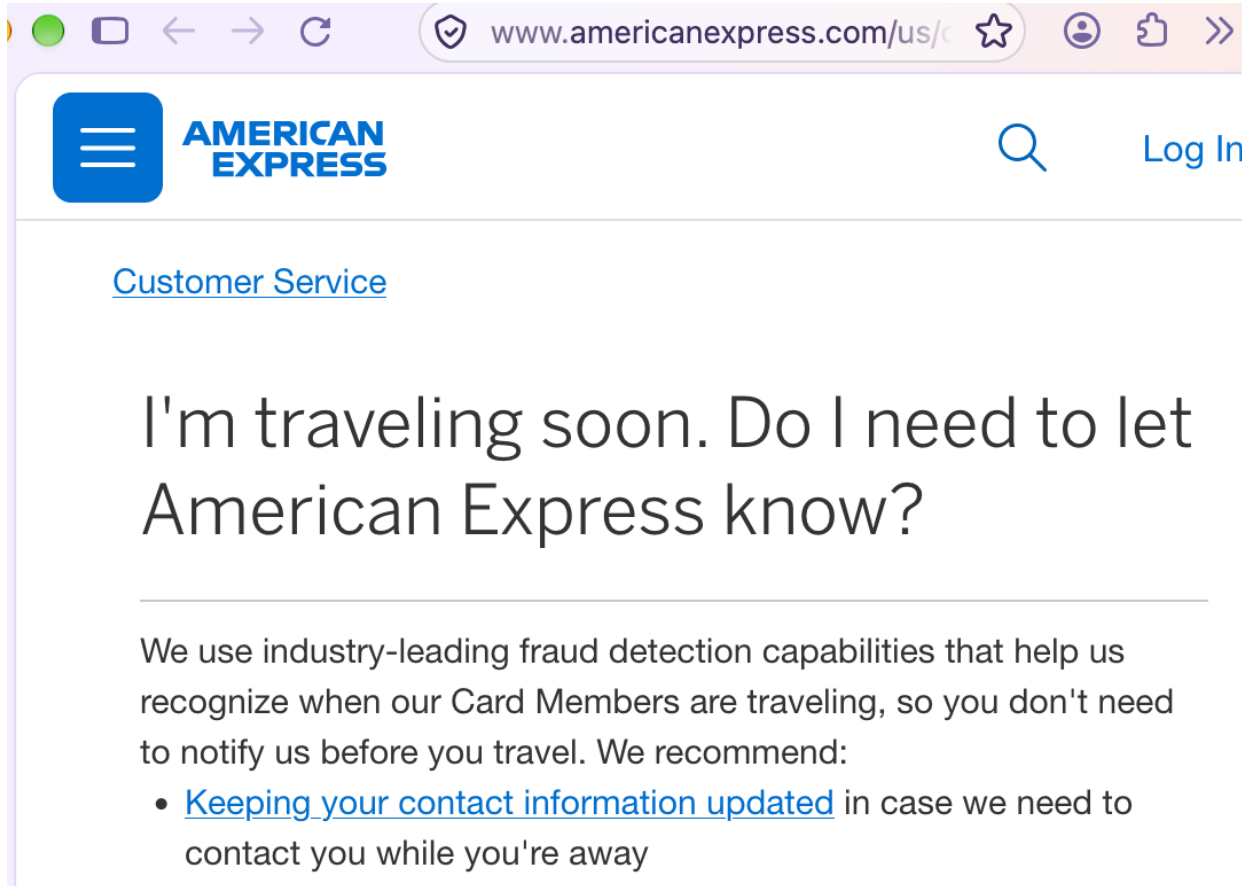
evidence:
some noise in the bushes



collecting/integrating more evidence...
it was probably just the wind



Application/Example: Fraud Detection



In the past, AE did collect travelling information.

- Why?
- Why isn't it needed anymore



Today's lecture

- Part 1: Two-step experiments and how to model them
- Part 2: Bayes-rule and how to go back from observations to latent causes
- Part 3: Applications: Credit Card fraud detection. Human probability judgment errors (self-study)

warm-up exercise

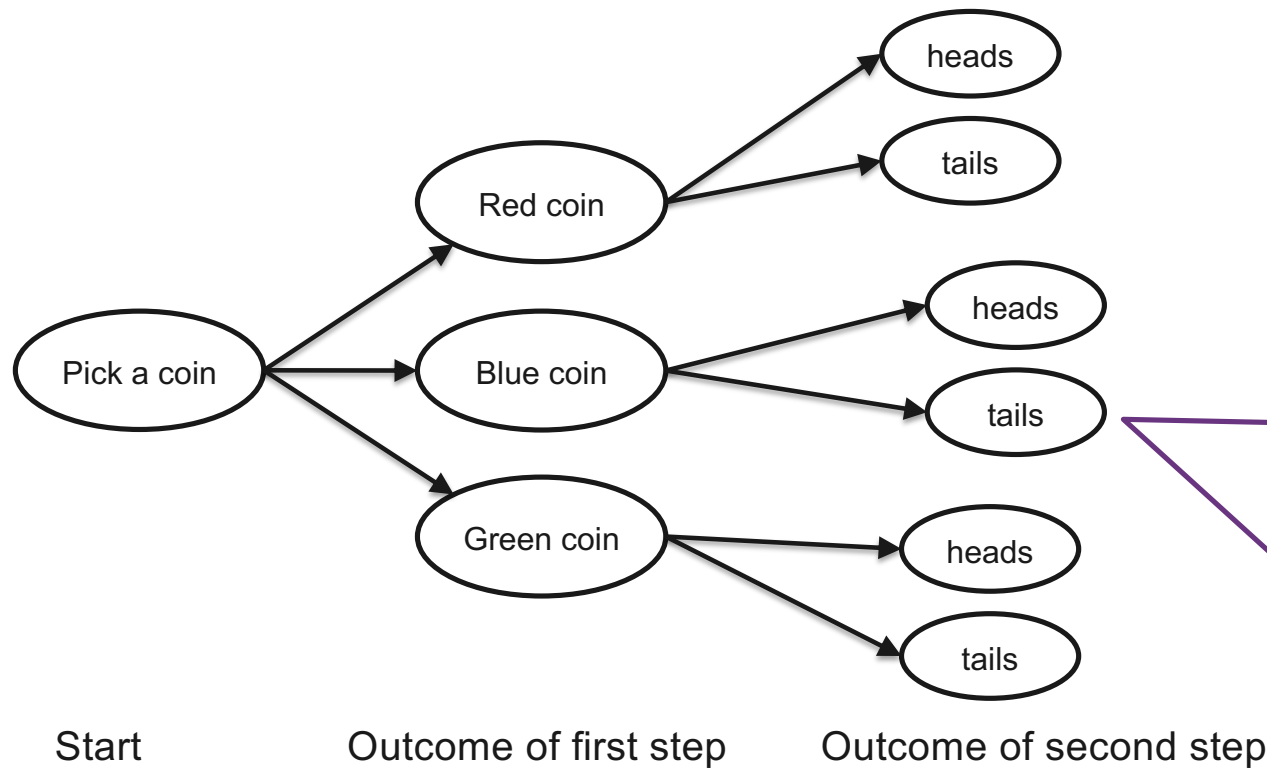
- repetition of basic probability rules and definitions
- in-class: 8 minutes now, together

A two-step experiment

- Consider a box with **three different coins**, a red, a blue and a green one. The red coin is fair. The others have different probabilities for head/tail:
 - Red coin R: $\Pr(\text{Side}=\text{heads}) = \Pr(S=\text{tails}) = 0.5$. (coin R is fair)
 - Blue coin B: $\Pr(S = \text{heads}) = 0.7$, $\Pr(S = \text{tails}) = 0.3$
 - Green coin G: $\Pr(S = \text{heads}) = 0.1$, $\Pr(S = \text{tails}) = 0.9$
- We now do a two-step experiment:
 - Step 1: you randomly pick one coin from the box.
 - Step 2: you toss the coin and observe the outcome, heads or tail.
- Question: What is the probability to observe "heads" ?
- Take a minute and think about this experiment. Can you calculate/guess the solution?

How to model two-step experiments: Tree diagram!

- A Tree Diagrams ...
 - is a **probabilistic model** that explains how data is generated (**generative model**).
 - shows how different random variables depend on each other.
 - serves as a structured visualization of the experiment.



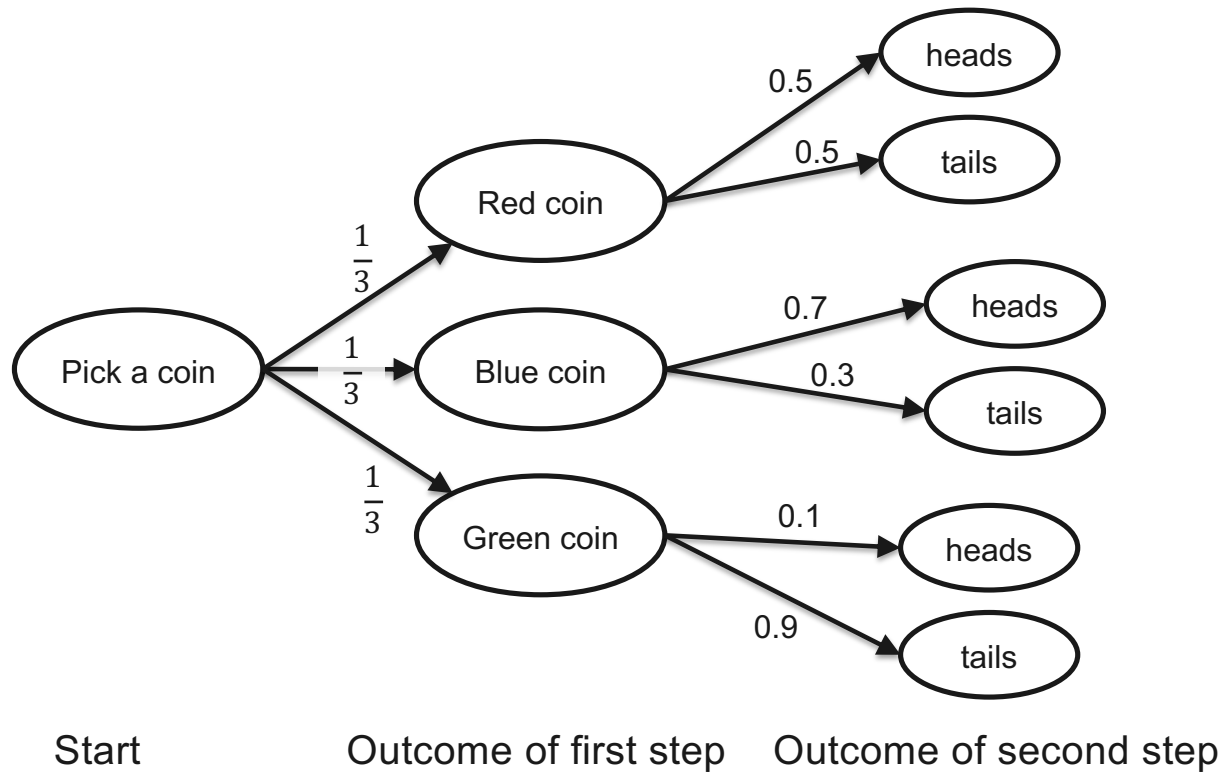
The tree encodes all (six) possible outcomes of the experiment and shows how each outcome was "generated".

For example, we reach this leaf of the tree if the first experiment resulted in "blue" **and** then the second in "tail":

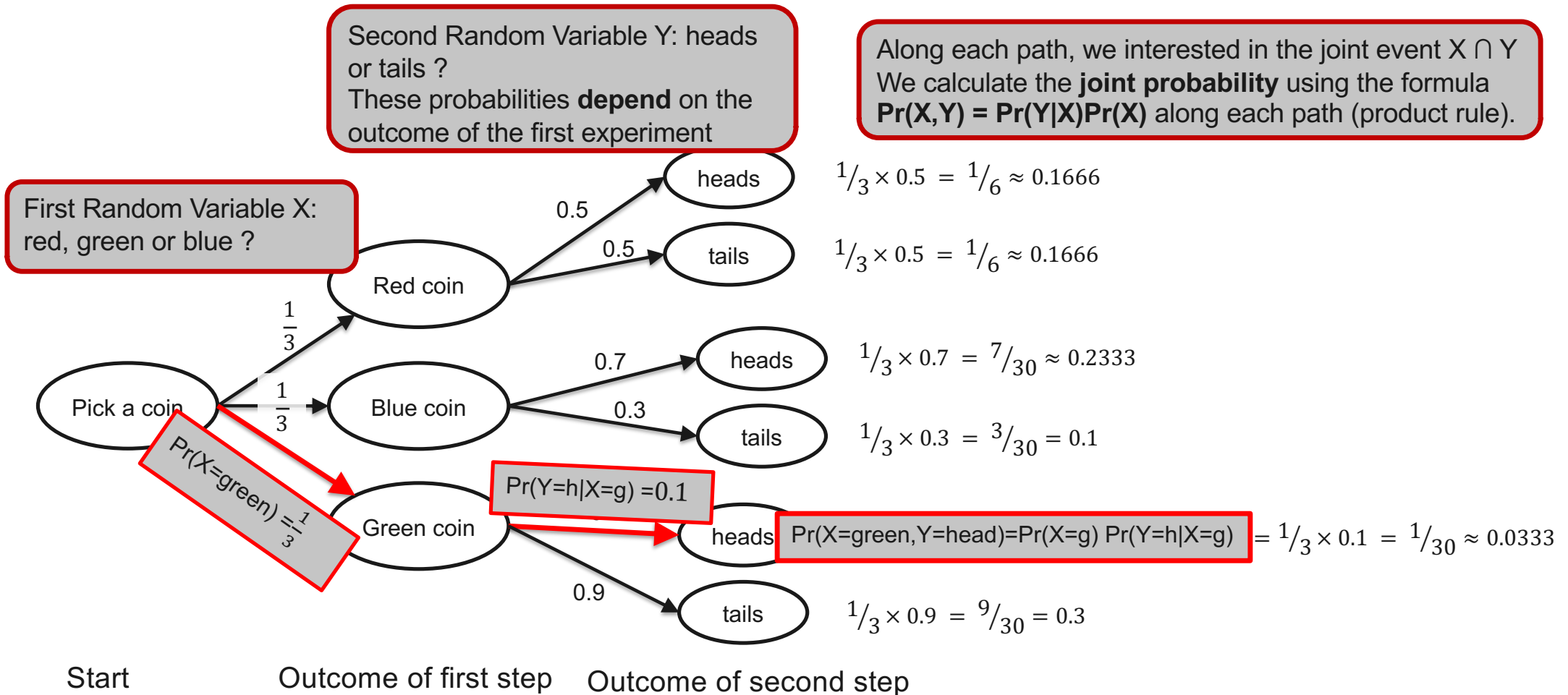
Coin=blue $\cap$ Side=tail

Tree diagrams

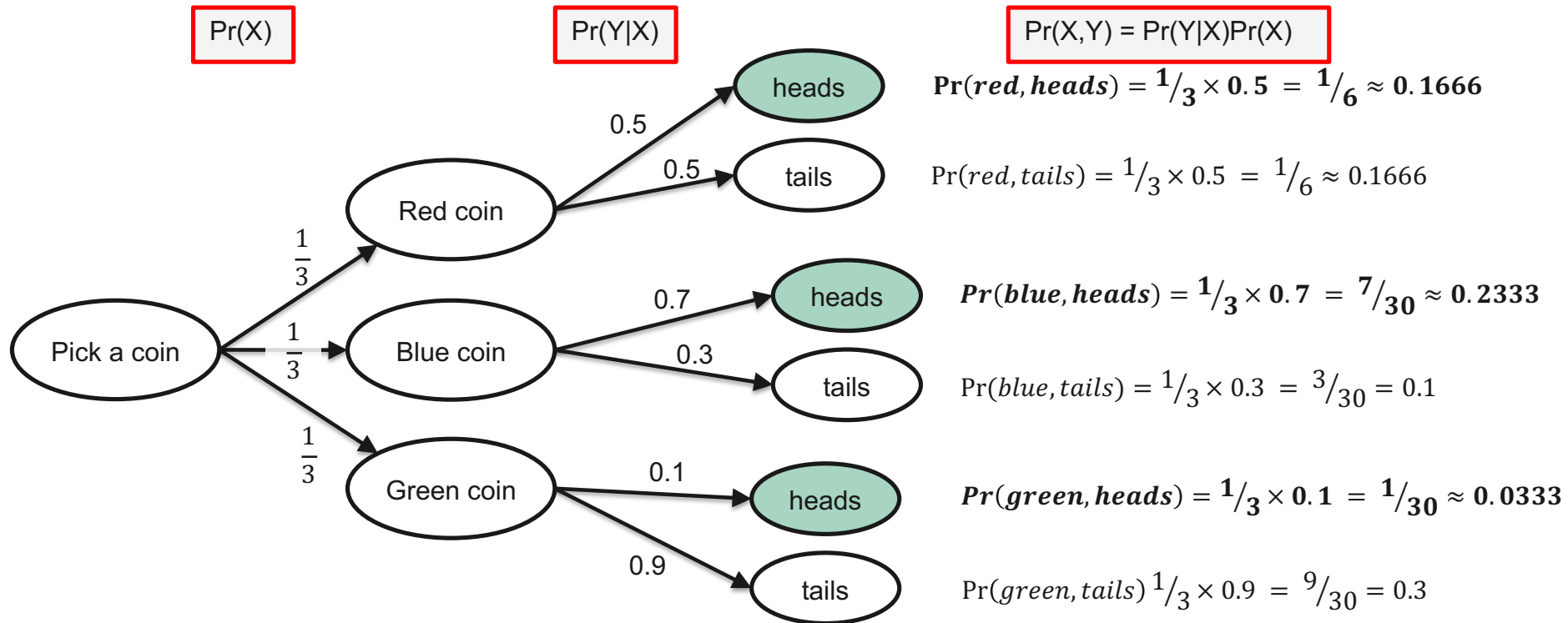
- Every arrow represents an event occurring with a certain probability
- We can assign the probability (from the problem statement) to each arrow:



Precise notation, and the math along a path.



Back to our question: What is the probability to observe "heads" ?



Observe: There are 3 possible realizations of the experiment (= 3 paths) that lead to the outcome 'heads'. To answer our question, we can sum their probabilities. The technical term for this is **marginalization**:

$$\Pr(Y = \text{heads}) = \sum_x \Pr(X = x, Y = \text{heads}) = \frac{1}{6} + \frac{7}{30} + \frac{1}{30} \approx 0.433$$

Same results shown in a table of Joint Probabilities

- The joint probability table shows the same result. Also, it becomes obvious how we marginalize to answer the initial question.
- Instead of drawing the tree diagram, we could have put the numbers and formulas directly in this form. But the tree diagram gives us a better understanding of the experiment and "what depends on what".

Joint Probability Table

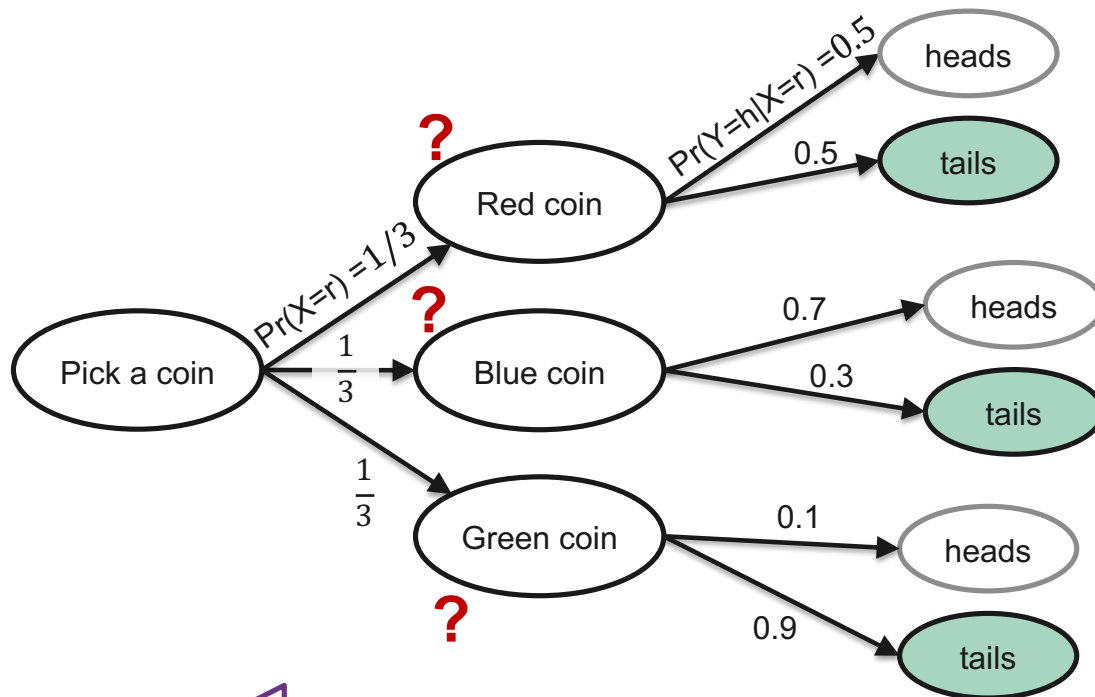
	Coin=red	Coin=blue	Coin=green	<i>Marginal Pr(Side)</i>
Side=head	Pr(red, head) $\approx$ 0.1666	0.2333	0.0333	0.433
Side=tail	Pr(red, tail) $\approx$ 0.1666	0.1	0.3	0.566
<i>Marginal Pr(Coin)</i>	0.333	0.333	0.333	1

Part 2: Bayes Rule.

- Tree diagrams are directed. We typically read them from "left to right" or "top to down". But what if we start from the end? Given the outcome of the second step, can we say something about the first step? Yes! In part 1, you learned how to model the generation of data. Now we learn how Bayes theorem lets us "invert" the direction of the model, that is, from observed outcomes (data), we can make probabilistic statements about the "hidden causes" of the observation.
- Simple example 1:
We observe tails? Which coin was drawn in step 1?
- Simple example 2:
We flip the same coin 3 times and observe tails, heads, heads. Which coin was drawn in step 1?
- Same math, different experiments:
 - A radar repeatedly measures a weak and noisy signal. What is the probability that an airplane is approaching?
 - A study investigates human learning from reward (Reinforcement Learning). In a controlled experiment, it observes a change in behaviour and asks: Did a specific memory in the brain influence the behaviour?
 - Credit Card Fraud Detection.
 - ... countless applications ...
- You learn how Bayes Theorem lets us answer such questions. These techniques have become a backbone in science and technology

Same experiment, different problem/question:

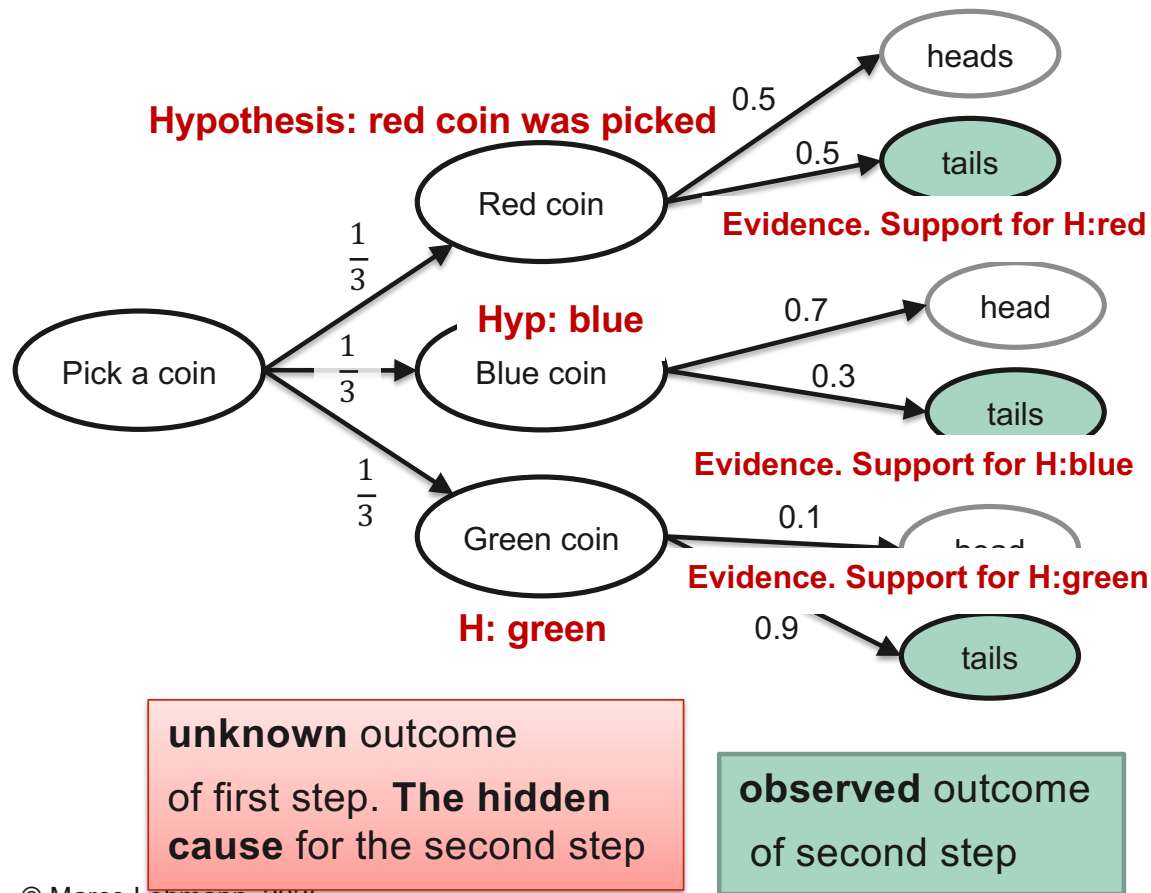
- We observe "tails". Which coin was drawn?



We start with the **observation** and a generative **model** (here in the form of an annotated tree). Then we "go back" to the **hidden cause** of the observation.

Application of Bayes Rule: Hypothesis testing

- We observe "tail". Which coin was drawn?



Given the stochastic nature of the process, we cannot answer the question.

But we can calculate the **probability** for each of the three options.

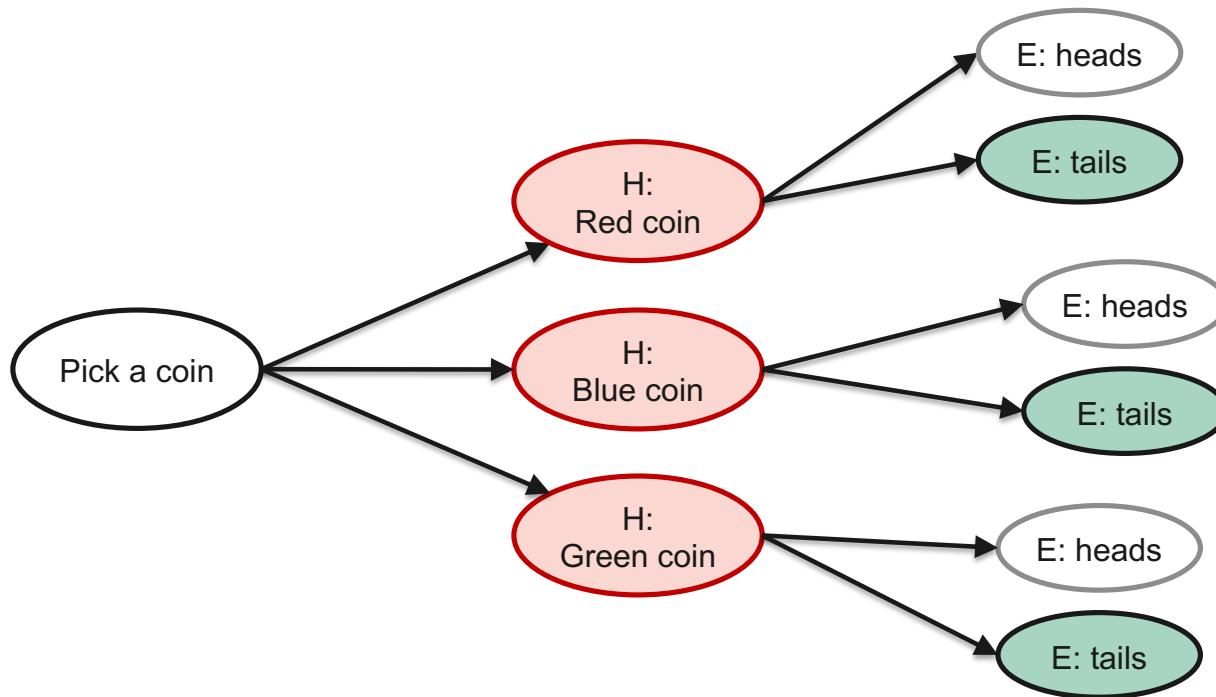
That is, we formulate three different **hypotheses H**. The graph captures our knowledge about how the data (measurement, observed outcome) was **generated**.

Then, for each hypothesis we ask: how **likely** is the outcome 'tail', if the hypothesis was true. The outcome is also sometimes called **Evidence E** (=support for a hypothesis).

Crucially, each hypothesis is supported not only by the observed outcome, but we also **integrate** our **prior knowledge** (our *knowledge before observing data*) In the absence of other information we assume that each hypothesis is equally likely ($\frac{1}{3}$ in this example).

Formulate the question in terms of probabilities

- We observe "tail". Which coin was drawn?

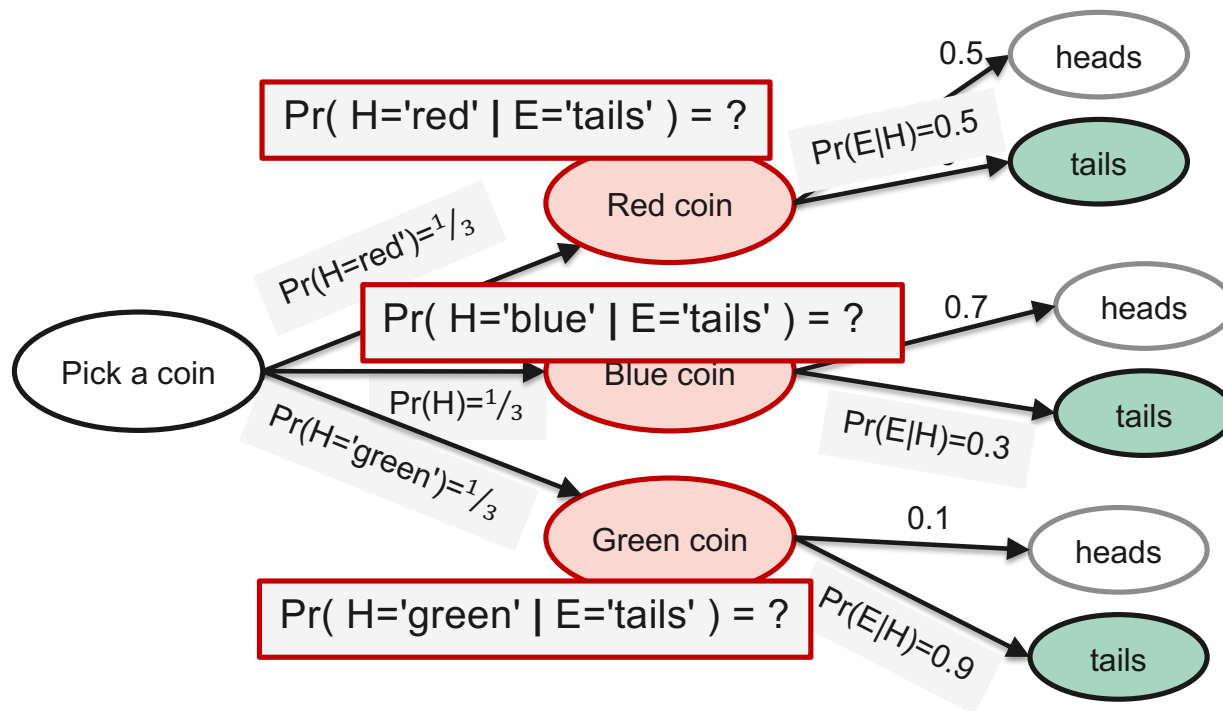


- Step 1: Formulate a question that asks for a probability. Use the correct, technical vocabulary:

- Step 2: formulate that question using mathematical notation:

Everything we know in one graph.

We can point to a node and ask a probabilistic question.



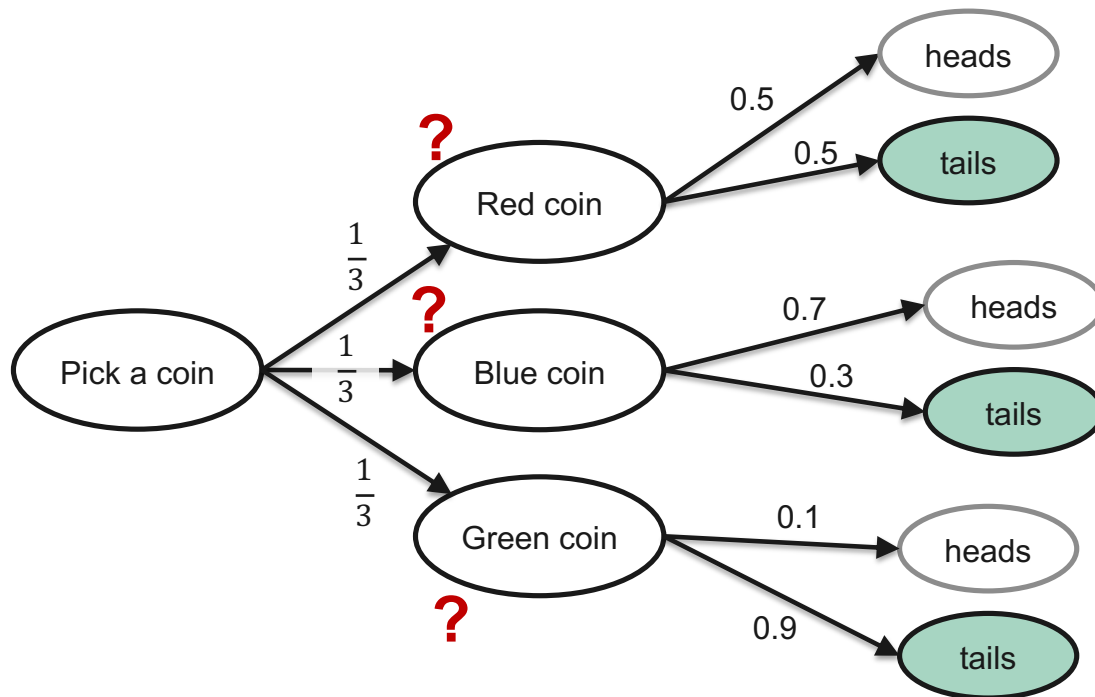
Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$\boxed{\text{Posterior}} = \frac{\boxed{\text{Likelihood}} \times \boxed{\text{Prior}}}{\boxed{\text{Normalizer}}}$$

With Bayes' rule, we invert the "direction"

- We observe "tail". Which coin was drawn?



$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

B|A is the "forward" direction of the **generative** model: cause -> observable
A|B is the inverted model: from observable to possible cause: this is also called **Bayesian inference**

How researcher describe this:

- "Bayes' Rule let's us invert the generative model"
- "We apply Bayesian inference"
- "We inferred the latent (hidden) cause using Bayes"

We start with the **observation** (B, Y, evidence, data, 2nd step) **and** a probabilistic generative **model** (here in the form of an annotated tree). Then we "go back" to the **hidden cause** (A, X, Hypothesis, 1st step) of the observation.

Bayes' Rule:

simple formula, but needs practice to see how to apply!

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(\mathbf{Hypothesis}|\mathbf{Evidence}) = \frac{P(\mathbf{Evidence}|\mathbf{Hypothesis}) \times P(\mathbf{Hypothesis})}{P(\mathbf{Evidence})}$$

$$P(\mathbf{H} = 'red' | \mathbf{E} = 'tails') = \frac{P(\mathbf{E} = 'tails' | \mathbf{H} = 'red') \times P(\mathbf{H} = 'red')}{P(\mathbf{E} = 'tails')}$$

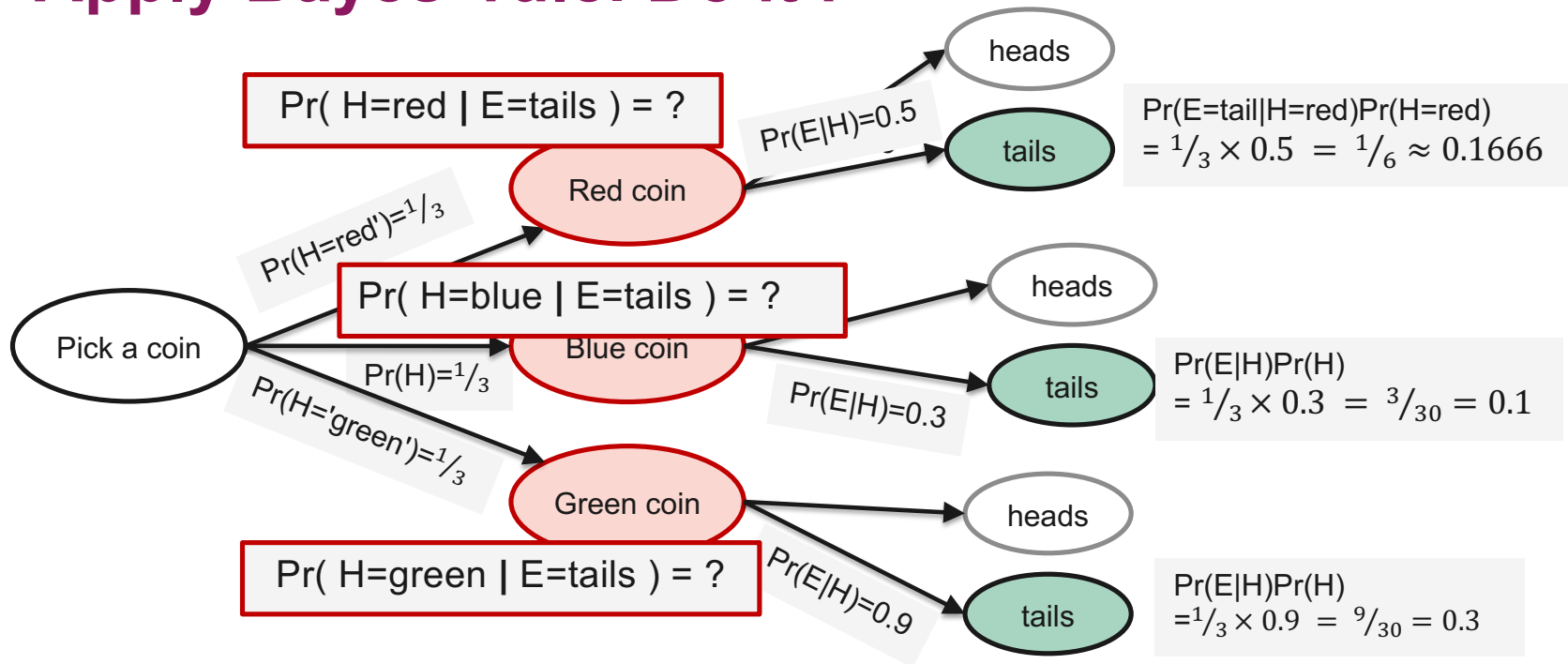
$$P(\mathbf{H} = 'blue' | \mathbf{E} = 'tails') = \dots$$

$$P(\mathbf{H} = 'green' | \mathbf{E} = 'tails') = \dots$$

Marginal Probability !

$$P(E) = \sum_H P(E|H)P(H)$$

Apply Bayes' rule. Do it !



- Here we use the results from the example in Part A.
- The denominator in Bayes formula is a normalization term. It is the same for all 3 expressions

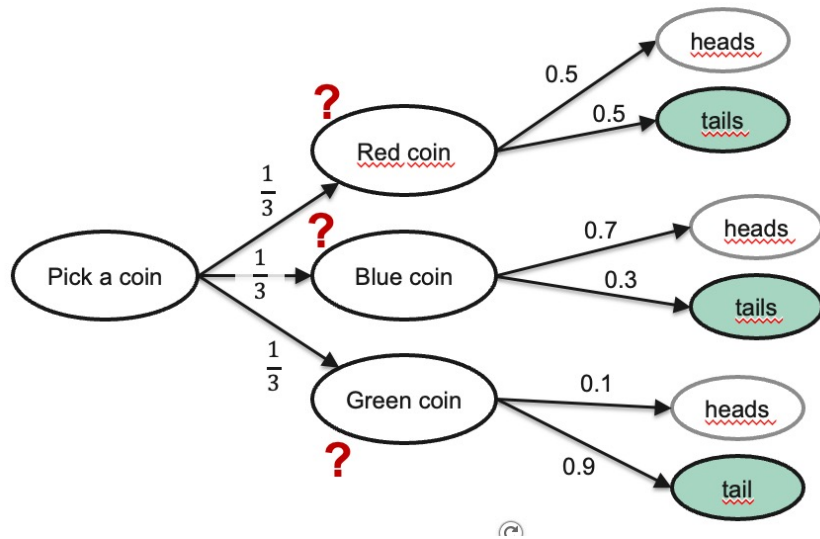
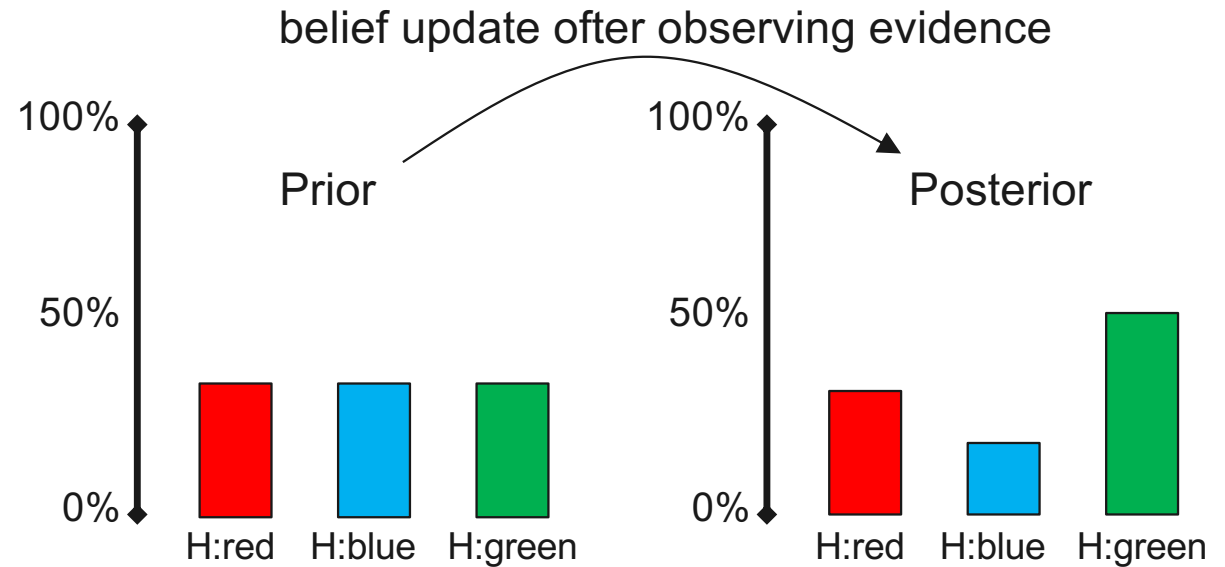
$$P(H = \text{'red'} | E = \text{'tails'}) = \frac{P(E = \text{'tails'} | H = \text{'red'}) \times P(H = \text{'red'})}{P(E = \text{'tails'})} = \dots \text{ (your calculation)}$$

$$P(H = \text{blue} | E = \text{tails}) =$$

$$P(H = \text{green} | E = \text{tails}) =$$

The result of bayesian inference: posterior distribution

- We observe "tails". Which coin was drawn?



$$P(H = red | E = tail) \approx 0.294$$

$$P(H = blue | E = tail) \approx 0.176$$

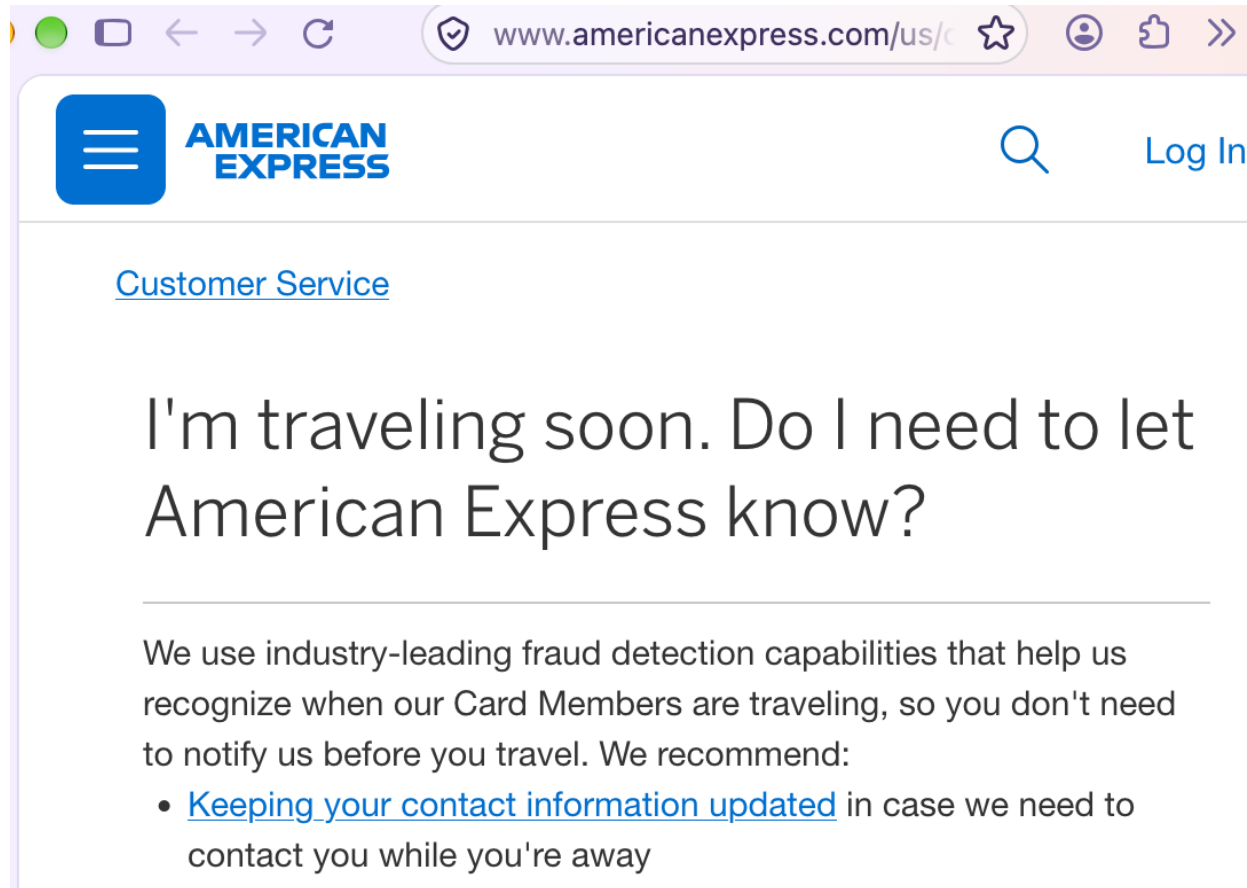
$$P(H = green | E = tail) \approx 0.529$$

The result is a distribution
(more precisely a PMF for the case of discrete random variables).

It is called the **Posterior Distribution**.

It is the result of **updating the Prior Distribution**
with the **evidence** (or data, or observation).

Application/Example: Fraud Detection

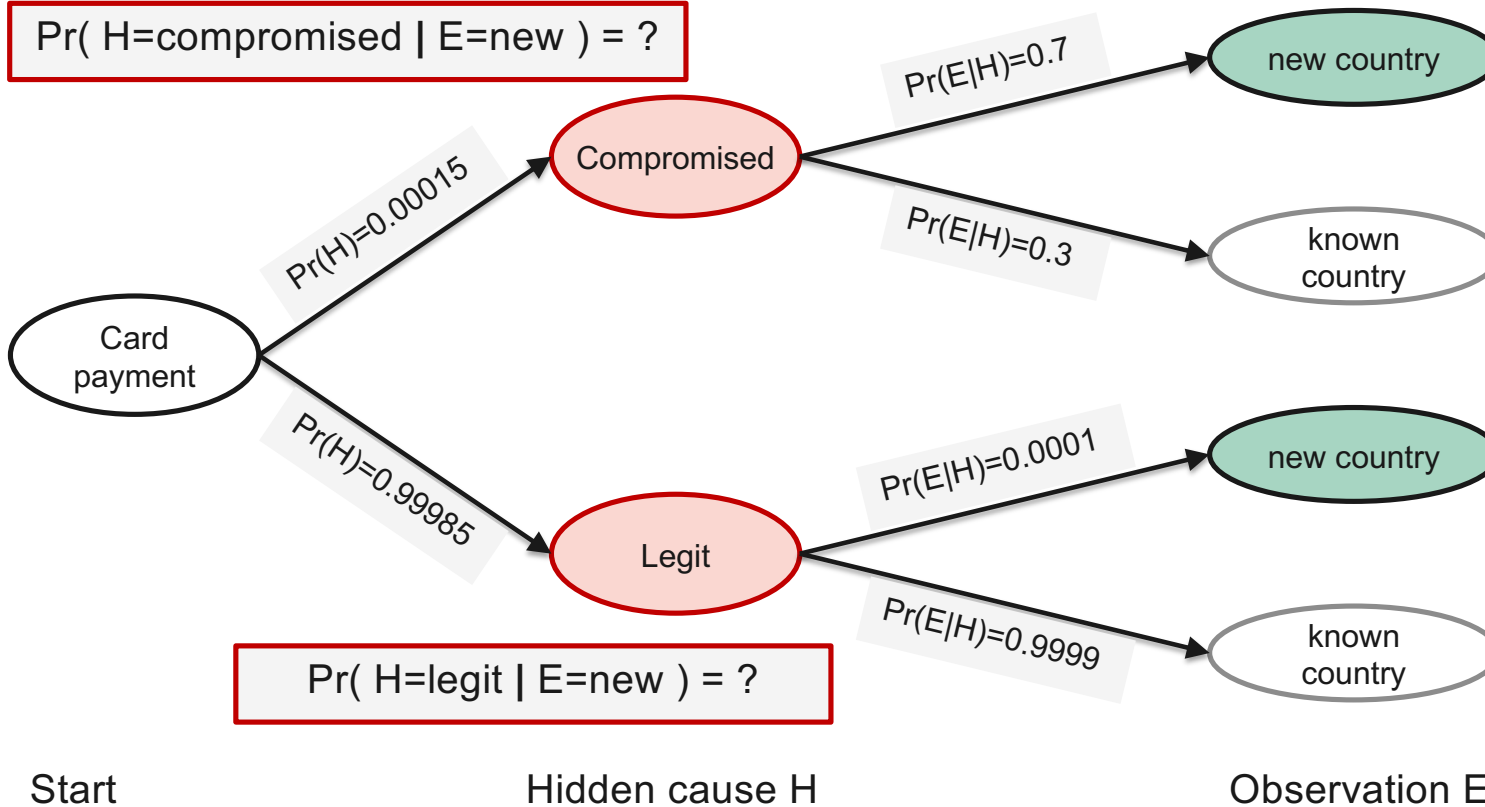


Fraud detection

- A card payment arrives from a country where this card was never used. Is it fraud? Should the bank block it?
- **Two-step model:**
 - Step 1, hidden cause H : the card is compromised or legit.
 - Step 2, observation E : the payment comes from a new country or from a known country.
- **What the bank knows:**
 - 0.015% of card payments are fraud. Prior: $\Pr(H=\text{compromised}) = 0.00015$
 - 70% of fraudulent payments come from a new country: $\Pr(E=\text{new} \mid H=\text{compromised}) = 0.7$
 - Legitimate payments from a new country depend on the customer:
 - Anna rarely travels. 1 in 10,000 of her payments: $\Pr(E=\text{new} \mid H=\text{legit}) = 0.0001$
 - For contrast, we also consider a 2nd customer: Ben travels for work. 1 in 200 of his payments: $\Pr(E=\text{new} \mid H=\text{legit}) = 0.005$
- **Question:** what is $\Pr(H=\text{compromised} \mid E=\text{new})$ for Anna? For Ben? Guess first.

Anna: everything we know in one tree

$$\Pr(H=\text{compromised} \mid E=\text{new}) = ?$$



$$\Pr(H=\text{legit} \mid E=\text{new}) = ?$$

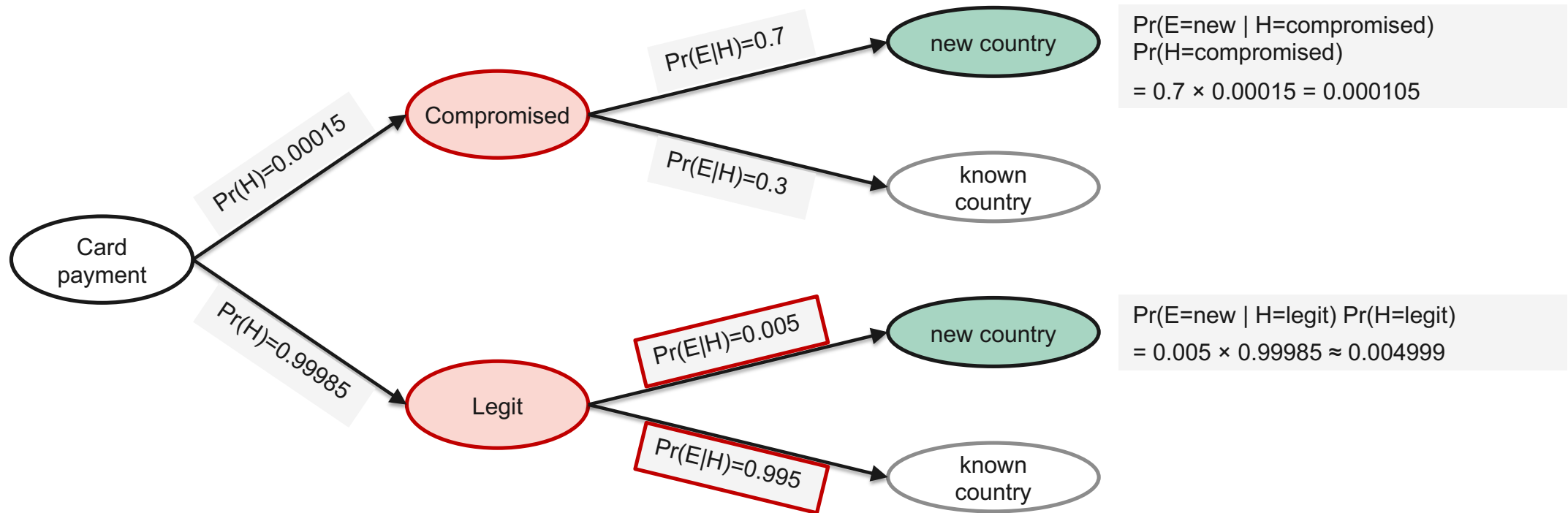
in-class:

Apply Bayes' rule.

Compute for Anna:

$\Pr(H=\text{compromised} \mid E=\text{new})$

Ben (travels often) : same payment, but different history

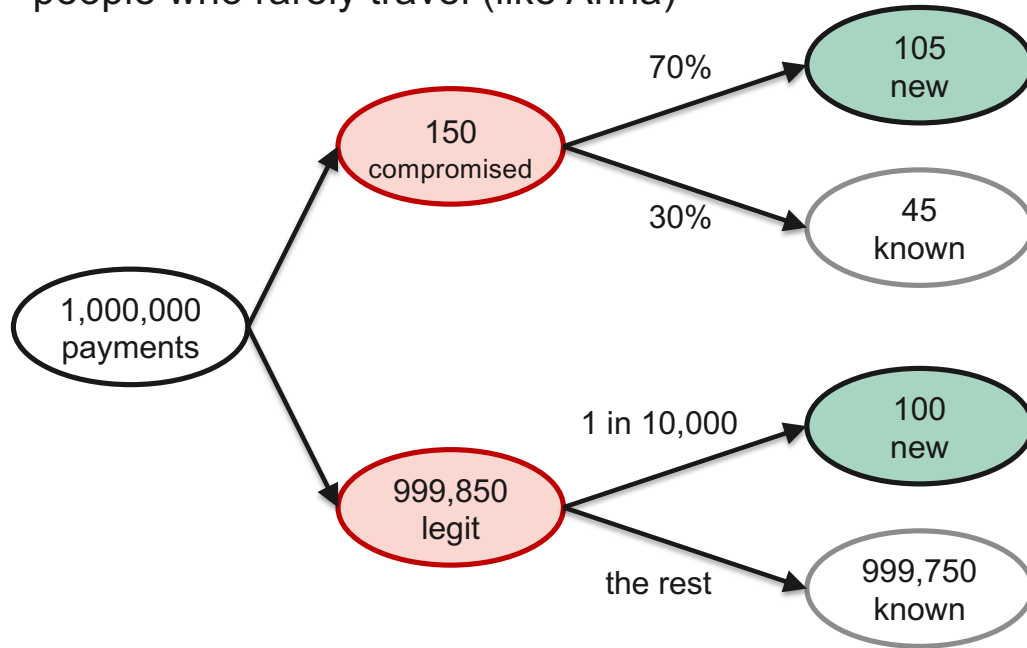


$$\Pr(H=\text{compromised} \mid E=\text{new}) = 0.000105 / (0.000105 + 0.004999) \approx 0.02$$

Only $\Pr(E=\text{new} \mid H=\text{legit})$ changed. The posterior drops from 0.51 (Anna) to 0.02 (Ben).

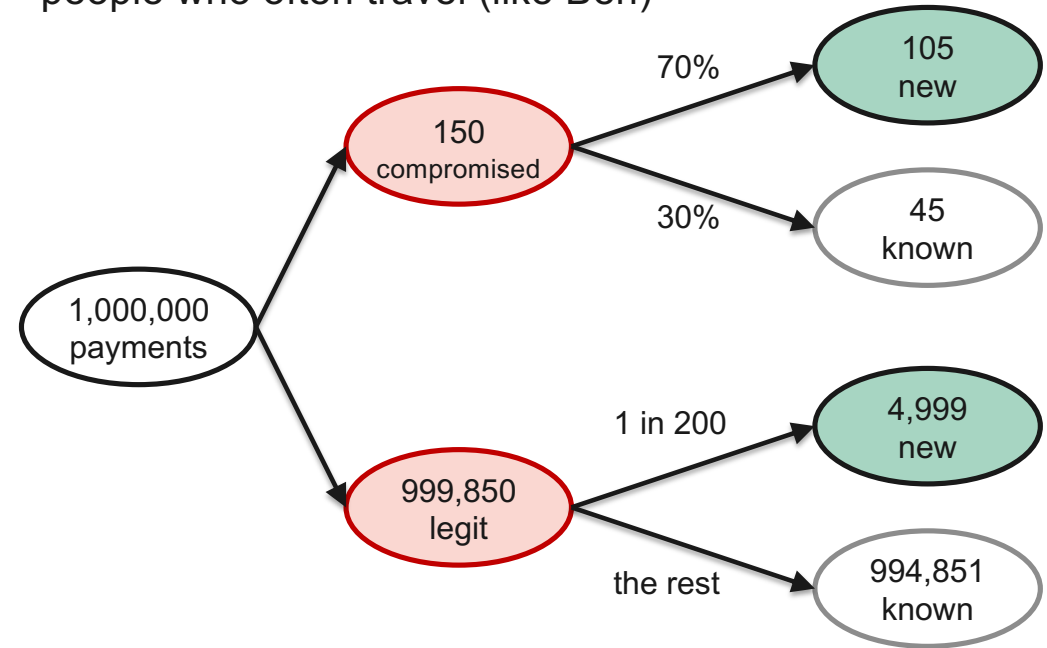
Check with frequencies: 1,000,000 payments

people who rarely travel (like Anna)



105 of 205 new-country payments: compromised card, 51%

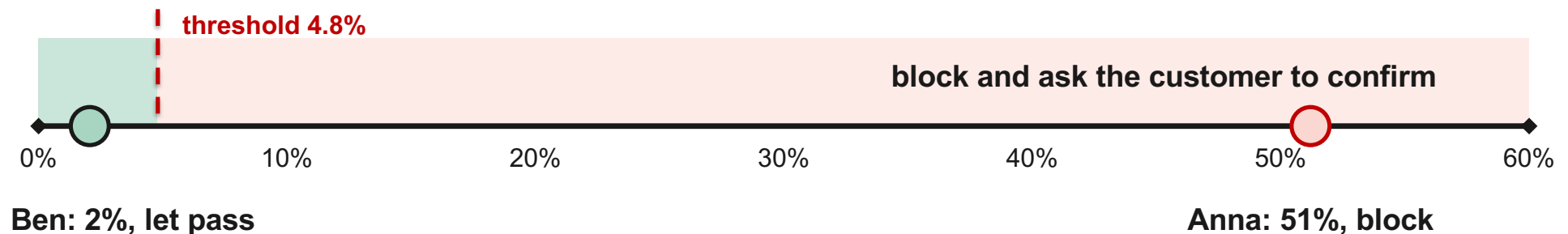
people who often travel (like Ben)



105 of 5,104 new-country payments: compromised card, 2%

From belief to decision: block or let pass?

- Bayes updates the probabilities. It does not tell us how to act! The costs decide the action.
- **Costs (assumptions):**
 - A fraud that passes costs the bank CHF 1,000.
 - Blocking a legitimate payment costs CHF 50: a customer stuck abroad, a call to the hotline, ...
- Block if the expected loss of letting the payment pass is larger:
 - $\Pr(H=\text{compromised} \mid E) \times 1,000 > \Pr(H=\text{legit} \mid E) \times 50$, that is $\Pr(H=\text{compromised} \mid E) > 50 / 1,050 \approx 4.8\%$



Can we do better?

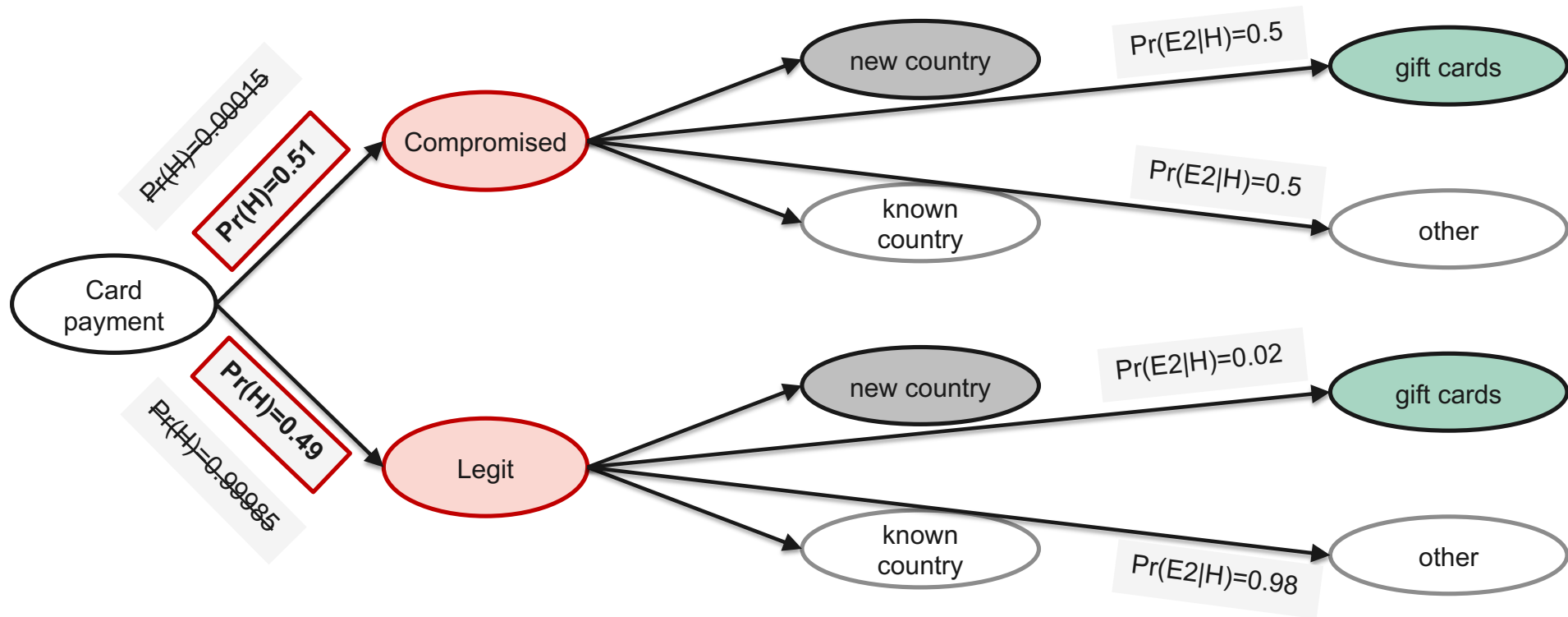
- Yes!
Bayes' rule lends itself to **integrating information sequentially**, as it arrives: the posterior after one observation is the prior for the next.

Anna: a second payment with the same card

- Before Anna confirms, the same card buys gift cards in an online shop. Is the card compromised? Should the bank block the card?
- **Same hidden cause, additional evidence:**
 - Hidden cause H : the card is compromised or legit. Same card, same H .
 - Observation E_1 , payment 1: new country.
 - Observation E_2 , payment 2: gift cards, or another purchase.
- **What the bank knows:**
 - Prior: the posterior after payment 1, $\Pr(H=\text{compromised}) = 0.51$
 - Half of the payments with compromised cards buy gift cards: $\Pr(E_2=\text{gift} \mid H=\text{compromised}) = 0.5$
 - Anna rarely buys gift cards. 1 in 50 of her payments: $\Pr(E_2=\text{gift} \mid H=\text{legit}) = 0.02$
- **Question:** what is $\Pr(H=\text{compromised} \mid E_1=\text{new}, E_2=\text{gift})$ for Anna? Guess first.

Assumptions for this example: the 0.5 and the 0.02. Given the state of the card, payment 2 does not depend on payment 1.

Sequential update (Bayesian filtering)



Start

Hidden cause H

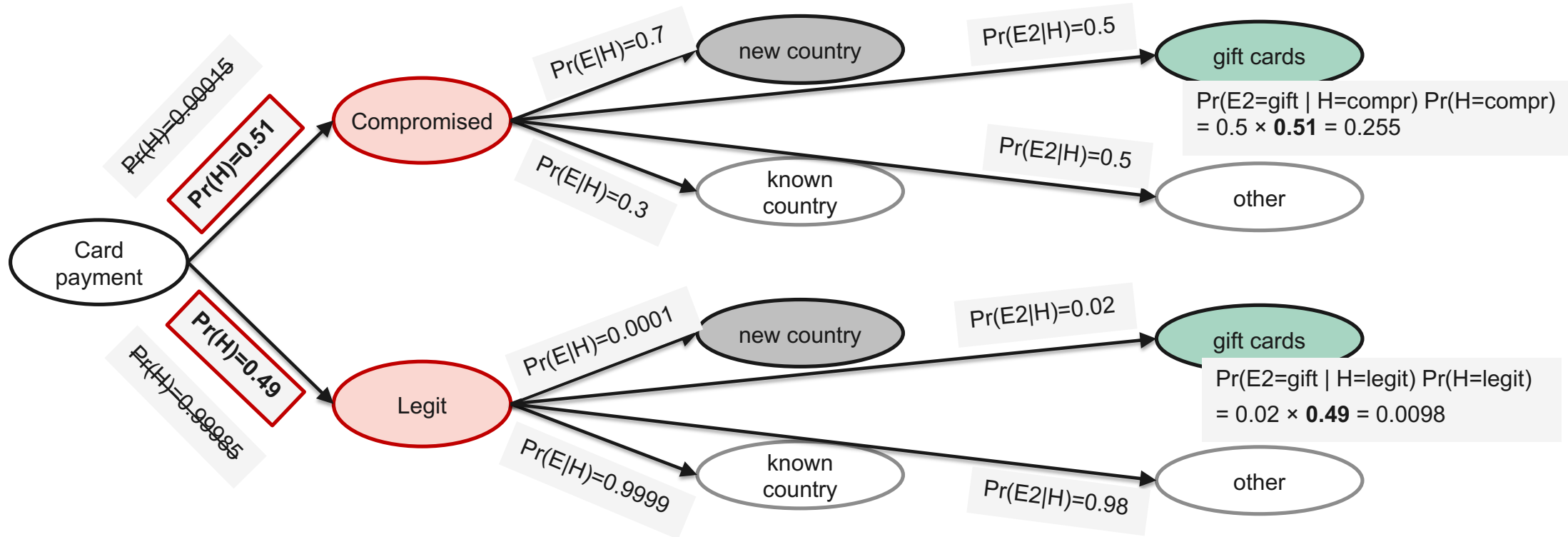
Observation E1

"new country" is observed
Prior is updated after first
evidence

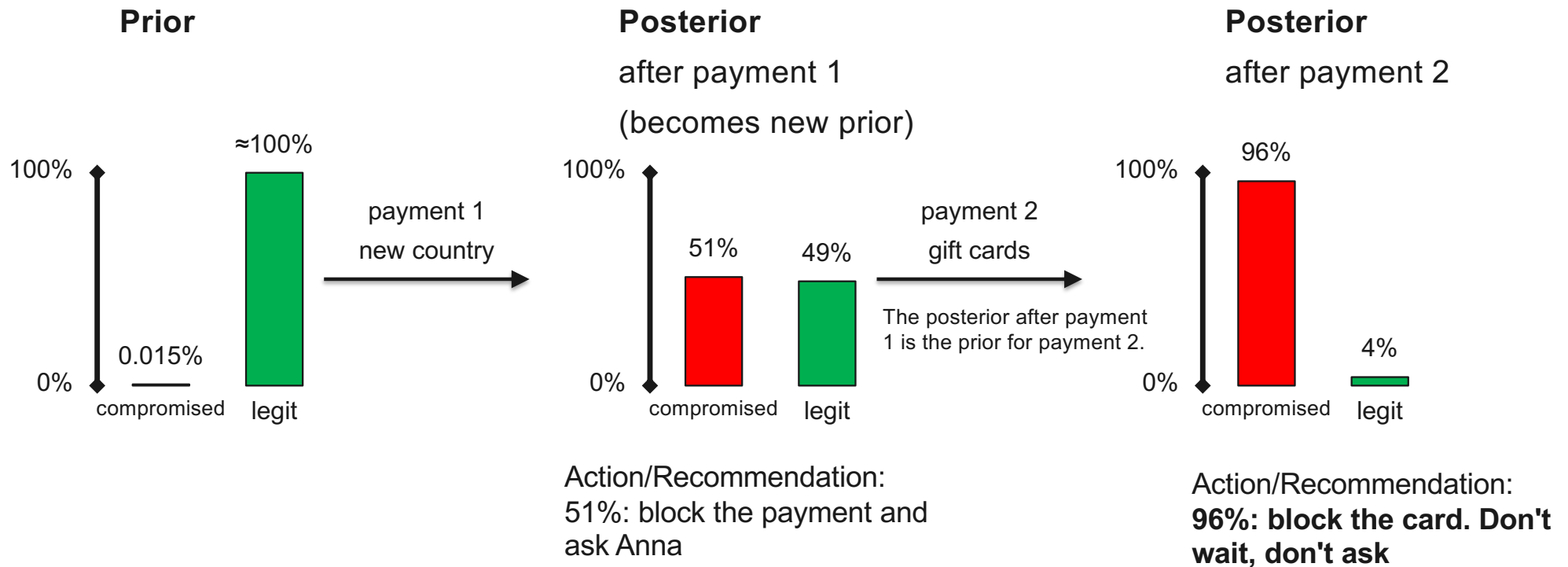
Observation E2

(collecting more evidence)
Do: calculate the posterior
after the second observation.

Sequential update (Bayesian filtering)



Sequential integration of evidence



Math/statistics are normative.

Humans are ... not always rational 🤔

- **Humans deviate from normative rationality.**
- Daniel Kahneman (especially in Thinking, Fast and Slow) highlights many **systematic fallacies and biases in human judgment** that arise from reliance on System 1 (fast, intuitive thinking).
- The Taxi-Cab Problem illustrates the **Base Rate Neglect** fallacy.

(if time) in-class exercise: solve the taxi-cab problem

- Describe the taxi-cab problem in a 2-step diagram
- Annotate all known probabilities using the notation seen in class.
- Apply Bayes' rule to calculate the posterior probability